Construction of measures with dilation

Henry de Thélin

Abstract

We give a construction of measures with partial sum of Lyapunov exponents bounded by below.

Key words: Lyapunov exponents, volume growth.
AMS: 28Dxx, 58F11.

Introduction

Let M be a compact C^1-Riemannian manifold of dimension d and let $f : M \mapsto M$ be a C^1-map.
For $1 \leqslant k \leqslant d$, we denote by $\mathcal{S}_k$ the set of C^1-maps $\sigma : D^k = [0,1]^k \mapsto M$. We define the k-volume of $\sigma \in \mathcal{S}_k$ with the formula:

$$V(\sigma) = \int_{D^k} |\Lambda^k T_x \sigma| d\lambda(x),$$

where $d\lambda$ is the Lebesgue measure on D^k and $|\Lambda^k T_x \sigma|$ is the norm of the linear map $\Lambda^k T_x \sigma : \Lambda^k T_x D^k \mapsto \Lambda^k T_{\sigma(x)} M$ induced by the Riemannian metric on M.
Some links between the volume growth of iterates of submanifolds of M and the entropy of f have been studied by Y. Yomdin (see [8] and [4]), S. E. Newhouse (see [7]), O.S. Kozlovski (see [6]) and J. Buzzi (see [2]).
In this article, we prove that the volume growth of iterates of submanifolds of M permits to create invariant measures with partial sum of Lyapunov exponents bounded by below.
More precisely, for $1 \leqslant k \leqslant d$ we define the k-dilation:

$$d_k := \limsup_{n \to \infty} \frac{1}{n} \log \sup_{\sigma \in \mathcal{S}_k} \frac{V(f^n \circ \sigma)}{V(\sigma)}.$$

We will prove the following theorem:

Theorem. *For all integer k between 1 and $d = dim(M)$ there exists an ergodic measure $\nu(k)$ for which:*

$$\sum_{i=1}^{k} \chi_i \geqslant d_k.$$

Here $\chi_1 \geqslant \chi_2 \geqslant \ldots \geqslant \chi_d$ are the Lyapunov exponents of $\nu(k)$.

Notice that when $k = d$ and f is a ramified covering in some sense, the theorem can be deduced from a result due to T.-C. Dinh and N. Sibony (see [3] paragraph 2.3).

Proof of the theorem

Let k be a positive integer between 1 and d. We have to prove that there exists an ergodic measure $\nu(k)$ for which

$$\sum_{i=1}^{k} \chi_i = \lim_{m\to\infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(k)(y) \geqslant d_k.$$

For the definition of Lyapunov exponents and for the statement of the previous equality, see [5] and [1] chapter 3.

There will be three steps in the proof of the theorem.
In the first one, we will change the dilation d_k into a dilation of $|\Lambda^k T_x f^n|$. More precisely, we will find points x_{n_l} with $\frac{1}{n_l} \log |\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant d_k - \varepsilon$.
In the second part, we will see that the dilation of $|\Lambda^k T_{x(n_l)} f^{n_l}|$ can be spread out in time. We will give the construction of a measure ν_l such that $d_k - 2\varepsilon \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y)$.
The third step of the proof will be to take the limit in the previous inequality.

1) First step

Let n_l be a subsequence such that:

$$\frac{1}{n_l} \log \sup_{\sigma \in \mathcal{S}_k} \frac{V(f^{n_l} \circ \sigma)}{V(\sigma)} \to d_k.$$

We can find now a sequence $\sigma_{n_l} \in \mathcal{S}_k$ which verifies:

$$\frac{1}{n_l} \log \frac{V(f^{n_l} \circ \sigma_{n_l})}{V(\sigma_{n_l})} \to d_k.$$

In the next lemma, we prove that we have dilation for $|\Lambda^k T_x f^n|$ for some x:

Lemma 1. *For all $l \geqslant 0$ there exists $x(n_l) \in M$ with:*

$$\log |\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant \log \left(\frac{V(f^{n_l} \circ \sigma_{n_l})}{2V(\sigma_{n_l})} \right).$$

Proof. Otherwise we would have an integer l such that for all $x \in M$:

$$|\Lambda^k T_x f^{n_l}| \leqslant \frac{V(f^{n_l} \circ \sigma_{n_l})}{2V(\sigma_{n_l})}.$$

So (see [1] chapter 3.2.3 for properties on exterior powers),

$$V(f^{n_l} \circ \sigma_{n_l}) = \int_{D^k} |\Lambda^k T_x (f^{n_l} \circ \sigma_{n_l})| d\lambda(x) = \int_{D^k} |\Lambda^k T_{\sigma_{n_l}(x)} f^{n_l} \circ \Lambda^k T_x \sigma_{n_l}| d\lambda(x)$$

is bounded by above by

$$\int_{D^k} |\Lambda^k T_{\sigma_{n_l}(x)} f^{n_l}||\Lambda^k T_x \sigma_{n_l})| d\lambda(x) \leqslant \frac{V(f^{n_l} \circ \sigma_{n_l})}{2}$$

and we obtain a contradiction.

□

Corollary 1. *There exists a sequence $\varepsilon(l)$ which converges to 0 such that:*

$$\frac{1}{n_l} \log |\Lambda^k T_{x(n_l)} f^{n_l}| \geqslant d_k - \varepsilon(l),$$

for some points $x(n_l)$ in M.

2) Second step

In this section, we will spread out in time the previous dilation.
Let m be a positive integer. We will now cut n_l with m different ways.
By using the Euclidian division, we can find q_l^i and r_l^i (for $i = 0, \dots, m-1$) such that:

$$n_l = i + m \times q_l^i + r_l^i$$

with $0 \leqslant r_l^i < m$.
If $i \in \{0, \dots, m-1\}$, we have:

$$|\Lambda^k T_{x(n_l)} f^{n_l}| \leqslant |\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}| \times \prod_{j=0}^{q_l^i - 1} |\Lambda^k T_{f^{i+jm}(x(n_l))} f^m| \times |\Lambda^k T_{x(n_l)} f^i|,$$

so, by using the previous corollary,

$$n_l(d_k - \varepsilon(l)) \leqslant \log |\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}| + \sum_{j=0}^{q_l^i - 1} \log |\Lambda^k T_{f^{i+jm}(x(n_l))} f^m| + \log |\Lambda^k T_{x(n_l)} f^i|.$$

If we take the sum on the m different ways to write n_l, we obtain:

$$mn_l(d_k-\varepsilon(l)) \leqslant \sum_{i=0}^{m-1} \log|\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}| + \sum_{i=0}^{m-1}\sum_{j=0}^{q_l^i-1} \log|\Lambda^k T_{f^{i+jm}(x(n_l))} f^m| + \sum_{i=0}^{m-1} \log|\Lambda^k T_{x(n_l)} f^i|.$$

We have to transform this estimate on a relation on a measure. To realize that, we remark that:

$$\log|\Lambda^k T_{f^p(x(n_l))} f^m| = \int \log|\Lambda^k T_y f^m| d\delta_{f^p(x(n_l))}(y),$$

where $\delta_{f^p(x(n_l))}$ is the dirac measure at the point $f^p(x(n_l))$.
So the previous inequality becomes:

$$d_k - \varepsilon(l) \leqslant a_l + \frac{1}{m}\int \log|\Lambda^k T_y f^m| d\left(\frac{1}{n_l}\sum_{i=0}^{m-1}\sum_{j=0}^{q_l^i-1} \delta_{f^{i+mj}(x(n_l))}\right)(y) + b_l$$

with

$$a_l = \frac{1}{mn_l}\sum_{i=0}^{m-1} \log|\Lambda^k T_{f^{i+mq_l^i}(x(n_l))} f^{r_l^i}|$$

and

$$b_l = \frac{1}{mn_l}\sum_{i=0}^{m-1} \log|\Lambda^k T_{x(n_l)} f^i|.$$

Now, because f is a C^1-map we have:

$$a_l \leqslant \frac{1}{mn_l}\sum_{i=0}^{m-1} \log L^{mk} \leqslant \frac{km^2}{mn_l}\log L$$

where $L = \max(\max_x |T_x f|, 1)$ and:

$$b_l \leqslant \frac{1}{mn_l}\sum_{i=0}^{m-1} \log L^{mk} \leqslant \frac{km^2}{mn_l}\log L.$$

So the sequences a_l and b_l are bounded by above by a sequence which converges to 0 when l goes to infinity.
In conclusion, we have:

$$d_k - \varepsilon'(l) \leqslant \frac{1}{m}\int \log|\Lambda^k T_y f^m| d\nu_l(y) \tag{1}$$

with

$$\nu_l = \frac{1}{n_l} \sum_{i=0}^{m-1} \sum_{j=0}^{q_l^i - 1} \delta_{f^{i+mj}(x(n_l))},$$

and $\varepsilon'(l)$ a sequence which converges to 0.

3) Third step

The aim of this section is to take a limit for ν_l in the equation (1).
First, observe that $\nu_l = \frac{1}{n_l} \sum_{p=0}^{n_l - m} \delta_{f^p(x(n_l))}$ and that the sequence $\frac{1}{n_l} \sum_{p=0}^{n_l - 1} \delta_{f^p(x(n_l))} - \nu_l$ converges to 0. In particular, there exists a subsequence of ν_l which converges to a measure ν which is a probability invariant under f and independant of m. We continue to call ν_l the subsequence which converges to ν. To complete the proof of the theorem, we have to take the limit in the equation (1). However, we have to be careful because the function $y \mapsto \log |\Lambda^k T_y f^m|$ is not continuous. But, we have the following lemma:

Lemma 2.

$$\limsup_{l \to \infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(y).$$

Proof. For $r \in \mathbb{N}$, let $\Phi_r(y) = \max(\log |\Lambda^k T_y f^m|, -r)$.
The functions Φ_r are continuous and the sequence Φ_r decreases to the map $y \mapsto \log |\Lambda^k T_y f^m|$ when r goes to infinity.
Then:

$$\frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \Phi_r(y) d\nu_l(y),$$

and,

$$\limsup_{l \to \infty} \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu_l(y) \leqslant \frac{1}{m} \int \Phi_r(y) d\nu(y)$$

because Φ_r is continuous. Now, we obtain the lemma by using the monotone convergence theorem.

$\square$

It remains to take the limit in the equation (1). We obtain then the

Corollary 2. *For all m, we have:*

$$d_k \leqslant \frac{1}{m} \int \log |\Lambda^k T_y f^m| d\nu(y).$$

In particular,

$$d_k \leqslant \int \sum_{i=1}^{k} \chi_i(y) d\nu(y)$$

where the $\chi_1 \geqslant \chi_2 \geqslant \ldots \geqslant \chi_d$ are the Lyapunov exponents of ν. Finally, by using the ergodic decomposition of ν, we obtain the existence of an ergodic measure $\nu(k)$ with:

$$d_k \leqslant \sum_{i=1}^{k} \chi_i.$$

References

[1] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer-Verlag, (1998).

[2] J. Buzzi, *Entropy, volume growth and Lyapunov exponents*, preprint (1996).

[3] T.-C. Dinh and N. Sibony, *Dynamique des applications d'allure polynomiale*, J. Math. Pures Appl., **82** (2003), 367-423.

[4] M. Gromov, *Entropy, homology and semialgebraic geometry*, Astérisque, **145-146** (1987), 225-240.

[5] A. Katok et B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encycl. of Math. and its Appl., vol. 54, Cambridge University Press, (1995).

[6] O.S. Kozlovski, *An integral formula for topological entropy of C^∞ maps*, Ergodic Theory Dynam. Systems, **18** (1998), 405-424.

[7] S. E. Newhouse, *Entropy and volume*, Ergodic Theory Dynam. Systems, **8** (1988), 283-299.

[8] Y. Yomdin, *Volume growth and entropy*, Israel J. Math., **57** (1987), 285-300.

Henry de Thélin
Université Paris-Sud (Paris 11)
Mathématique, Bât. 425
91405 Orsay
France